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*Curves of Triple Curvature.**

JAMES G. HARDY.

The present paper is written to bring together and, in some measure, to add to the results which have been obtained concerning curves L of triple curvature. Equations of motion for systems in a four-dimensional space have been deduced and used to introduce the notion of an instantaneous plane of rotation. The derivation of the equations of motion is not new but it seemed best for the sake of clearness to retain it. By constructing the principal tetrahedroid at a point of a curve of triple curvature and studying its motion by means of the kinematical equations obtained, geometrical interpretations of the six rotations and also a set of formulæ corresponding to the Serret-Frenet formulæ for curves of double curvature have been arrived at. These formulæ have been applied to the study of curves L and, in particular, of the osculating hypersphere and the locus of its centers.

NOTE. Several of the results of this paper were contained in a paper read by me in 1898 before the Mathematical Seminary of the Johns Hopkins University. The articles by Professor Lovett and Mr. Hatzidakis in the American Journal of Mathematics, vol. XXII, have appeared since then.

THE MOTION OF SYSTEMS IN A SPACE OF FOUR DIMENSIONS.

Velocities in an invariable system.

1. To discuss the motion of a system in a space of four dimensions we proceed as in the discussion of a similar problem in three dimensional space. The motion will be referred to a set of four rectangular axes OX , OY , OZ , OW

* Conf. Brunel, Mathematische Annalen, XIX, p. 48 ; Pirondini, Giornali di Matematiche, XXVIII, p. 237 ; Piccioli, Giornali di Matematiche, XXXVI, p. 273.

through a fixed point O . Imagine a second set of four rectangular axes, having the same orientation as the first, invariably connected with the moving system, and whose origin o is one of its points. To determine the position of the system at any instant it is sufficient to know, in terms of the time, the coordinates X_0, Y_0, Z_0, W_0 of the origin o , and the nine direction cosines of the moving axes with respect to the fixed axes. Denote the moving axes by ox, oy, oz, ow ; coordinates referred to the fixed axes by X, Y, Z, W , and with respect to the moving axes by x, y, z, w . The moving axes form a moving "tetrahedroid" and the six planes passed through these axes in pairs form a moving "hexahedron." The sixteen direction cosines of the one set of axes referred to the other are given by the following table:

	x	y	z	w
X	α_1	β_1	γ_1	δ_1
Y	α_2	β_2	γ_2	δ_2
Z	α_3	β_3	γ_3	δ_3
W	α_4	β_4	γ_4	δ_4

Consider first the motion of an invariable system, that is, a system whose points do not move relatively to each other, and let x, y, z, w be the coordinates of a point P of the system referred to the moving axes, and X, Y, Z, W be the coordinates of the same point referred to the fixed axes; then

$$\begin{aligned}
 X &= X_0 + \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 w, \\
 Y &= Y_0 + \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 w, \\
 Z &= Z_0 + \alpha_3 x + \beta_3 y + \gamma_3 z + \delta_3 w, \\
 W &= W_0 + \alpha_4 x + \beta_4 y + \gamma_4 z + \delta_4 w.
 \end{aligned} \tag{1}$$

Differentiating these equations for t , we have the projections of the velocity V of the point P on the fixed axes,

$$\begin{aligned}
 V_x &= \frac{dX_0}{dt} + x \frac{d\alpha_1}{dt} + y \frac{d\beta_1}{dt} + z \frac{d\gamma_1}{dt} + w \frac{d\delta_1}{dt}, \\
 V_y &= \frac{dY_0}{dt} + x \frac{d\alpha_2}{dt} + y \frac{d\beta_2}{dt} + z \frac{d\gamma_2}{dt} + w \frac{d\delta_2}{dt}, \\
 &\dots\dots\dots \\
 &\dots\dots\dots
 \end{aligned} \tag{2}$$

the x, y, z, w being constant in an invariable system.

Projecting V_x, V_y, V_z, V_w on the axes ox, oy, oz, ow , we have, as the projections V_x, V_y, V_z, V_w of the velocity V on the moving axes,

$$\begin{aligned} V_x &= \alpha_1 V_x + \alpha_2 V_y + \alpha_3 V_z + \alpha_4 V_w, \\ V_y &= \beta_1 V_x + \beta_2 V_y + \beta_3 V_z + \beta_4 V_w, \\ &\dots\dots\dots \\ &\dots\dots\dots \end{aligned} \quad (3)$$

These expressions for V_x, V_y, V_z, V_w become simple if we substitute in the second members of (3) the values of V_x, V_y, V_z, V_w given in (2), and remember that such sums as

$$\alpha_1 \frac{d\alpha_1}{dt} + \alpha_2 \frac{d\alpha_2}{dt} + \alpha_3 \frac{d\alpha_3}{dt} + \alpha_4 \frac{d\alpha_4}{dt}$$

are zero, and if we use the following abbreviations:

$$\begin{aligned} p_{12} &= \sum \beta_\kappa \frac{d\alpha_\kappa}{dt} = - \sum \alpha_\kappa \frac{d\beta_\kappa}{dt}, \\ p_{13} &= \sum \alpha_\kappa \frac{d\gamma_\kappa}{dt} = - \sum \gamma_\kappa \frac{d\alpha_\kappa}{dt}, \\ p_{23} &= \sum \gamma_\kappa \frac{d\beta_\kappa}{dt} = - \sum \beta_\kappa \frac{d\gamma_\kappa}{dt}, \\ p_{14} &= \sum \delta_\kappa \frac{d\alpha_\kappa}{dt} = - \sum \alpha_\kappa \frac{d\delta_\kappa}{dt}, \\ p_{24} &= \sum \delta_\kappa \frac{d\beta_\kappa}{dt} = - \sum \beta_\kappa \frac{d\delta_\kappa}{dt}, \\ p_{34} &= \sum \delta_\kappa \frac{d\gamma_\kappa}{dt} = - \sum \gamma_\kappa \frac{d\delta_\kappa}{dt}, \end{aligned} \quad (4)$$

We then find as the expressions of the projections of the velocity on the moving axes,

$$\left. \begin{aligned} V_x &= V_x^0 - p_{14}w + p_{13}z - p_{12}y, \\ V_y &= V_y^0 + p_{13}x - p_{24}w - p_{23}z, \\ V_z &= V_z^0 + p_{23}y - p_{13}x - p_{34}w, \\ V_w &= V_w^0 + p_{34}z + p_{24}y + p_{14}x, \end{aligned} \right\} \quad (5)$$

where $V_x^0, V_y^0, V_z^0, V_w^0$ denote the projections of the velocity V of the origin of the moving axes on the moving axes,

$$\left. \begin{aligned} V_x^0 &= \alpha_1 \frac{dX_0}{dt} + \alpha_2 \frac{dY_0}{dt} + \alpha_3 \frac{dZ_0}{dt} + \alpha_4 \frac{dW_0}{dt}, \\ &\dots\dots\dots \end{aligned} \right\} \quad (6)$$

2. The interpretation of equations (5) is as follows : They show that the velocity V of each point P of the system is the geometric sum of two vectors V and T ; the first being the same for all points of the body and equal and parallel to the velocity of the moving origin ; and the second, T , differing for different parts of the body and having as its projections on the moving axes

$$\begin{aligned} -p_{14}w + p_{13}z - p_{12}y, \\ p_{12}x - p_{24}w - p_{23}z, \\ p_{23}y - p_{13}x - p_{34}w, \\ p_{34}z + p_{24}y + p_{14}x. \end{aligned}$$

The vector T is the velocity the point P would possess if the only motion were one of rotation about a plane through the moving origin ; the quantities p_{ij} being the projections of the rotation on the six planes of the moving hexahedron, and the four quantities just written being its projections on the four axes.

This rotation may be called the instantaneous rotation, and we say that the velocity of any point in a body is the resultant of a velocity of translation equal to the velocity of any point o of the body, and a velocity of rotation about a plane through o .

Instantaneous plane of rotation.

3. For convenience, I shall speak of any relation between x, y, z, w as the equation of a hypersurface ; two relations as the equation of a surface ; three relations as the equation of a line. An expression of the form

$$ax + by + cz + dw + e = 0 \tag{7}$$

will be said to represent a hyperplane ; two such relations, a plane.

If an invariable system moves in a space of four dimensions, and we wish to find those points in it having at any instant a minimum velocity, we form from (5) the expression

$$V^2 = A^2 + B^2 + C^2 + D^2, \tag{8}$$

where A, B, C, D are the right-hand members of (5), and equate the four derivatives with respect to x, y, z, w to zero. We find

$$\left. \begin{aligned} p_{12}B - p_{13}C + p_{14}D &= 0, \\ -p_{12}A + p_{23}C + p_{24}D &= 0, \\ p_{13}A - p_{23}B + p_{34}D &= 0, \\ -p_{14}A - p_{24}B - p_{34}C &= 0, \end{aligned} \right\} \tag{9}$$

which, solved for x, y, z, w , give the points sought. To get the minimum velocity itself, we solve (9) for x, y, z, w and substitute their values in (8). The system (9) being linear and homogeneous in A, B, C, D , we will divide each equation in it through by D and solve the first three of the resulting equations for the ratios of A, B, C to D . This supposes that all the quantities A, B, C, D are not zero; that is, that the velocity V is not zero. We find

$$\frac{A}{D} = \left| \begin{array}{ccc} -p_{14} & p_{12} & -p_{13} \\ -p_{24} & 0 & p_{23} \\ -p_{34} & -p_{23} & 0 \end{array} \right| \div \left| \begin{array}{ccc} 0 & p_{12} & -p_{13} \\ -p_{12} & 0 & p_{23} \\ p_{13} & -p_{23} & 0 \end{array} \right|; \quad (10)$$

a determinant whose denominator is a skew symmetric determinant of odd order and, therefore, zero. Then the numerator

$$\left| \begin{array}{ccc} -p_{14} & p_{12} & -p_{13} \\ -p_{24} & 0 & p_{23} \\ -p_{34} & -p_{23} & 0 \end{array} \right| = -(p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}) \quad (11)$$

must vanish; that is, *in order that there be in a moving system a set of points with minimum velocity not zero, the relation*

$$(p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}) = 0 \quad (12)$$

must be satisfied between the six quantities p_{ij} .

The determinant in (11) being the determinant of the first three equations (9) divided by D and it being zero, and the equations being non-homogeneous, it follows that these equations are not independent and (9) *represents a plane*, the locus of all points whose velocity is minimum, *called the central plane of motion*. The motion may be looked on as made up of a gliding along the central plane and a rotation about it. No matter which three equations (9) are taken, their determinant is zero.

If the velocity at any instant *is* zero, that is, if the motion reduce entirely to a rotation, we must have

$$V^2 = A^2 + B^2 + C^2 + D^2 = 0;$$

therefore

$$A = B = C = D = 0,$$

or

$$\left. \begin{array}{l} V_x^0 - p_{14}w + p_{13}z - p_{12}y = 0, \\ V_y^0 + p_{12}x - p_{24}w - p_{23}z = 0, \\ V_z^0 + p_{23}y - p_{13}x - p_{34}w = 0, \\ V_w^0 + p_{34}z + p_{24}y + p_{14}x = 0. \end{array} \right\} \quad (13)$$

These equations (13) have as their determinant

$$\begin{vmatrix} 0 & , & p_{12} & , & -p_{13} & , & p_{14} \\ -p_{12} & , & 0 & , & p_{23} & , & p_{24} \\ p_{13} & , & -p_{23} & , & 0 & , & p_{34} \\ -p_{14} & , & -p_{24} & , & -p_{34} & , & 0 \end{vmatrix},$$

which, being a skew-symmetric determinant of even order, may be written

$$(p_{12} p_{34} + p_{13} p_{24} + p_{14} p_{23})^2.$$

The expression in parenthesis is the first member of (12) and if we take

$$(p_{12} p_{34} + p_{13} p_{24} + p_{14} p_{23}) = 0,$$

the four equations (13) reduce to two which *represent a plane called the instantaneous plane of rotation*, it being the locus of all points having at the instant considered a velocity equal to zero.

When the moving system has one point fixed we may take the fixed point for the origin of both the fixed and the moving axes; the equations (13) reduce to

$$\left. \begin{aligned} -p_{14}w + p_{13}z - p_{12}y &= 0, \\ p_{12}x - p_{24}w - p_{23}z &= 0, \\ p_{23}y - p_{13}x - p_{34}w &= 0, \\ p_{34}z + p_{24}y + p_{14}x &= 0. \end{aligned} \right\} \quad (14)$$

a set of four linear, homogeneous equations with four unknowns. If these are to be satisfied by a set of values of x, y, z, w not all zero, it is necessary that their determinant, which is exactly the determinant written after equations (13) vanish. In this case the four equations reduce to two and we have an instantaneous plane passing through the fixed point.

If in equations (13) we make

$$V_w^0 = w = p_{14} = p_{24} = p_{34} = 0,$$

the resulting equations

$$\begin{aligned} V_x^0 + p_{13}z - p_{12}y &= 0, \\ V_y^0 + p_{12}x - p_{23}z &= 0, \\ V_z^0 + p_{23}y - p_{13}x &= 0 \end{aligned}$$

are the equations of the instantaneous axis of rotation of an invariable system moving in a space of four dimensions.

Velocities in any system.

4. Suppose now that the moving system is not an invariable system, but that its points are free to move among themselves. Then, instead of treating x, y, z, w as constants in differentiating equations (1), we treat them as functions of the time t , and equations (5) become

$$\begin{aligned} V_x &= V_x^0 - p_{14}w + p_{13}z - p_{12}y + \frac{dx}{dt}, \\ V_y &= V_y^0 + p_{12}x - p_{24}w - p_{23}z + \frac{dy}{dt}, \\ V_z &= V_z^0 + p_{23}y - p_{13}x - p_{34}w + \frac{dz}{dt}, \\ V_w &= V_w^0 + p_{34}z + p_{24}y + p_{14}x + \frac{dw}{dt}, \end{aligned} \tag{15}$$

which are the projections of the velocity of a moving point in a moving system upon a set of axes invariably connected with the moving system.

Expressions for the rotations in terms of the direction cosines.

5. As in case of rotations in a space of three dimensions (see Darboux, I, p. 4), the six quantities p_{ij} may be expressed in terms of the sixteen direction cosines $\alpha_1, \dots, \delta_4$ and their derivatives with respect to the time. (See *American Journal of Mathematics*, vol. 20, p. 138.) Suppose, constructed through a fixed point of space, a tetrahedroid T' with the same arrangement of axes as in the one T fixed in the moving system, and let the axes of T' always remain parallel to those of T . Then, as far as the rotation is concerned, the motion of T' is the same as that of T , and the motion of T' will be given by equations (15) when the terms $V_x^0, \dots, V_w^0$ are made zero. Take a point on the fixed axis OX at a distance of unity from the origin; its coordinates referred to T' are $\alpha_1, \beta_1, \gamma_1, \delta_1$. Its velocity being zero, equations (14) give

$$\begin{aligned} \frac{d\alpha_1}{dt} &= p_{12}\beta_1 - p_{13}\gamma_1 + p_{14}\delta_1, \\ \frac{d\beta_1}{dt} &= -p_{12}\alpha_1 + p_{23}\gamma_1 + p_{24}\delta_1, \\ \frac{d\gamma_1}{dt} &= p_{13}\alpha_1 - p_{23}\beta_1 + p_{34}\delta_1, \\ \frac{d\delta_1}{dt} &= -p_{14}\alpha_1 - p_{24}\beta_1 - p_{34}\gamma_1. \end{aligned} \tag{16}$$

Similarly we may get three other sets of equations like the above, except that the subscript 1 is replaced by 2, 3, 4 respectively.

Making, successively, the subscript 1, 2, 3, 4 in the first equation (16), multiplying the resulting equations respectively by $\beta_1, \beta_2, \beta_3, \beta_4$, and adding, we find

$$\sum_1^4 \beta_\kappa \frac{d\alpha_\kappa}{dt} = \sum p_{12}\beta_\kappa^2 - \sum p_{13}\beta_\kappa\gamma_\kappa - \sum p_{14}\beta_\kappa\delta_\kappa,$$

or

$$p_{12}dt = \sum \beta_\kappa d\alpha_\kappa.$$

Similarly

$$\begin{aligned} p_{12}dt &= \sum \beta d\alpha = -\sum \alpha d\beta; & p_{23}dt &= \sum \gamma d\beta = -\sum \beta d\gamma, \\ p_{13}dt &= \sum \alpha d\gamma = -\sum \gamma d\alpha; & p_{24}dt &= \sum \delta d\beta = -\sum \beta d\delta, \\ p_{14}dt &= \sum \delta d\alpha = -\sum \alpha d\delta; & p_{34}dt &= \sum \delta d\gamma = -\sum \gamma d\delta, \end{aligned}$$

a set of equations expressing the rotations in terms of the sixteen direction cosines and their derivatives with respect to the time. They are analogous to those for a space of three dimensions. (Darboux, I, p. 5.)

APPLICATION TO CURVES OF TRIPLE CURVATURE.

The principal directions.

6. When a line is such that five consecutive points on it are not in general situated in the same ordinary space, it is said to be a *line of triple curvature*. Two consecutive points determine the *tangent* at any point; three consecutive points determine the *ordinary osculating plane* and four consecutive points determine the *osculating hyperplane*. The straight line situated in the osculating plane and perpendicular to the tangent is called the *principal normal* at the point M of tangency. The straight line contained in the osculating hyperplane and perpendicular to the tangent and the principal normal is called the *principal binormal* at M ; and a fourth line perpendicular to the three just defined, and, consequently, to the osculating hyperplane also, is called the *trinormal* at M . The binormal and trinormal are so called because they are respectively perpendicular to two and three consecutive tangents. The four lines just defined determine what we call the *principal directions* at any point M of a curve of triple curvature, and the tetrahedroid T formed by them at M is called the *principal tetrahedroid* at the point. T is fixed to the curve, and as M moves along L the tetrahedroid T moves in space. Its motion is given by the kinematical equations which precede, and by its use formulæ may be found for curves L corresponding to the Serret-Frenet formulæ for curves of double curvature.

The three curvatures.

7. The limit of the ratio of the angle of two consecutive tangents to the arc of the curve included between their points of contact is the first curvature at the point considered, and is written

$$\lim_{s \rightarrow 0} \left[\frac{d\phi_1}{ds} \right] = \frac{1}{\rho_1} \equiv \text{first curvature.}$$

If, through the origin of the fixed axes, we draw lines k of length 1 parallel to the tangents of the curve L , their extremities will trace out a spherical curve Γ on a sphere of unit radius having its center at o , and the coordinates of the extremity of any one of these lines are the direction cosines of the corresponding tangent to L . The angle between two consecutive lines k is the angle between two consecutive tangents of L , and is measured by the arc of Γ included between its sides. Then, by reason of the above definition of the first curvature, we may write

$$\frac{1}{\rho_1} = \sqrt{\frac{d\alpha_1^2}{ds^2} + \frac{d\alpha_2^2}{ds^2} + \frac{d\alpha_3^2}{ds^2} + \frac{d\alpha_4^2}{ds^2}}.$$

The limit of the ratio of two consecutive osculating planes to the arc between the point of contact is the second curvature of the curve considered:

$$\lim_{s \rightarrow 0} \left[\frac{d\phi_2}{ds} \right] = \frac{1}{\rho_2} = \text{second curvature.}$$

Making use of a method similar to the one just used for getting the second expression for the first curvature, we may write

$$\frac{1}{\rho_2} = \sqrt{\frac{d\gamma_1^2}{ds^2} + \frac{d\gamma_2^2}{ds^2} + \frac{d\gamma_3^2}{ds^2} + \frac{d\gamma_4^2}{ds^2}}.$$

The limit of the ratio of the angle of two consecutive trinormals to the arc ds is the third curvature of the point considered:

$$\lim_{s \rightarrow 0} \left[\frac{d\phi_3}{ds} \right] = \frac{1}{\rho_3} = \text{third curvature.}$$

We also have

$$\frac{1}{\rho_3} = \sqrt{\frac{d\delta_1^2}{ds^2} + \frac{d\delta_2^2}{ds^2} + \frac{d\delta_3^2}{ds^2} + \frac{d\delta_4^2}{ds^2}}.$$

Interpretation of the quantities p_{ij} .

8. If, at any ordinary point M of a curve L , we form the tetrahedroid T and take the tangent as the axis of x , the principal normal as the axis of y , the principal binormal as the axis of z and the trinormal as the axis of w , we have a system of moving rectangular axes, with the origin at M , to which points on the curve may be referred. Let x, y, z, w be the coordinates of any point referred to these moving axes, and X, Y, Z, W the coordinates of the same point referred to a set of rectangular axes, fixed in space, having an arbitrary point o as origin. Take the arc s of the curve as the independent variable, or, what is the same thing, suppose

$$\frac{ds}{dt} = 1.$$

Then

$$V_x^0 = 1, \quad V_y^0 = V_z^0 = V_w^0 = 0;$$

and also,

$$\alpha_1 = \frac{dx}{ds}, \quad \alpha_2 = \frac{dy}{ds}, \quad \alpha_3 = \frac{dz}{ds}, \quad \alpha_4 = \frac{dw}{ds}, \quad (17)$$

From equations (16) we find

$$\left. \begin{aligned} d\alpha_\kappa &= (p_{12}\beta_\kappa - p_{13}\gamma_\kappa + p_{14}\delta_\kappa) ds, \\ d\beta_\kappa &= (-p_{12}\alpha_\kappa + p_{23}\gamma_\kappa + p_{24}\delta_\kappa) ds, \\ d\gamma_\kappa &= (p_{13}\alpha_\kappa - p_{23}\beta_\kappa + p_{34}\delta_\kappa) ds, \\ d\delta_\kappa &= (-p_{14}\alpha_\kappa - p_{24}\beta_\kappa - p_{34}\gamma_\kappa) ds. \end{aligned} \right\} \quad (18)$$

Since the principal binormal, whose direction cosines are $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, is perpendicular to two consecutive tangents, whose direction cosines are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3, \alpha_4 + d\alpha_4$, it follows that

$$\sum_1^4 (\alpha_\kappa \gamma_\kappa) = 0 \quad \text{and} \quad \sum \gamma_\kappa (\alpha_\kappa + d\alpha_\kappa) = 0.$$

The first of these is true on account of the orthogonality of the axes; and, if we remember that

$$\sum \gamma_\kappa d\alpha_\kappa = -p_{13}ds,$$

the second will give the condition

$$\sum \gamma_\kappa d\alpha_\kappa = -p_{13}ds = 0,$$

and, therefore,

$$p_{13} = 0.$$

This corresponds to the condition $q = 0$ for ordinary space.

If, in the same way, we express the condition that the trinormal is perpendicular to three consecutive tangents, we find the additional necessary relation

$$\Sigma \delta_{\kappa} (\alpha_{\kappa} + d\alpha_{\kappa}) = 0, \text{ or } \Sigma \delta_{\kappa} d\alpha_{\kappa} = p_{14} ds = 0,$$

and, therefore,

$$p_{14} = 0.$$

Two consecutive osculating hyperplanes of a curve of triple curvature intersect in an osculating plane; then the principal normal to L at M , since it lies in the osculating plane, lies also in both osculating hyperplanes and is therefore perpendicular to two consecutive trinormals. Expressing this fact we have

$$\Sigma \beta_{\kappa} (\delta_{\kappa} + d\delta_{\kappa}) = 0, \text{ or } \Sigma \beta_{\kappa} d\delta_{\kappa} = -p_{24} ds = 0,$$

which requires

$$p_{24} = 0.$$

Of the six rotations p_{ij} three are zero, as we have seen, when the tetrahedroid T is connected with the curve as above described. In order to interpret the remaining three it is necessary to use the definitions given in (7) of the three curvatures of a line L .

9. Construct through some fixed point a tetrahedroid T_1 whose axes are parallel to those of T . We get in this way a tetrahedroid whose motion, so far as rotation is concerned, is exactly that of T . A point at the distance 1 from the origin on the x -axis of T_1 has a velocity whose projections (see equations (5)) are 0, p_{12} , 0, 0; that is to say, this point describes a path $p_{12} ds$; or, what is the same thing, the tangent to the curve turns through an angle $p_{12} ds$ while the point of contact describes the arc ds . Consequently

$$\text{the first curvature} = \frac{1}{\rho_1} = \frac{p_{12} ds}{ds} = p_{12}.$$

The projections of the velocity of a point on the principal binormal at a distance 1 from the origin are 0, $-p_{23}$, 0, p_{34} . Therefore,

$$\begin{aligned} \text{the second curvature} &= \frac{1}{\rho_2} = \frac{\sqrt{p_{34}^2 + (-p_{23})^2} \cdot ds}{ds}, \\ \frac{1}{\rho_2} &= \sqrt{p_{34}^2 + (-p_{23})^2}. \end{aligned}$$

A point at the distance 1 from the origin and on the trinormal has a velocity whose projections are 0, 0, $-p_{34}$, 0; therefore,

$$\text{the third curvature} = \frac{1}{\rho_3} = \frac{-p_{34} ds}{ds} = -p_{34}.$$

These last three formulæ give us the desired interpretation for the quantities p_{12} , p_{23} , p_{34} ;

$$\begin{aligned} p_{12} &= \frac{1}{\rho_1}, \\ p_{23} &= -\sqrt{\frac{1}{\rho_2^2} - \frac{1}{\rho_3^2}} \equiv \frac{1}{H}, \\ p_{34} &= -\frac{1}{\rho_3}, \end{aligned} \tag{19}$$

When $\rho_3 = \infty$ these reduce immediately to the corresponding formulæ for ordinary space.

“Serret-Frenet” Formulæ.

10. If we introduce the values given by (19) into (18) and also use $p_{13} = p_{14} = p_{24} = 0$, we find

$$\begin{aligned} \frac{d\alpha_\kappa}{ds} &= \frac{\beta_\kappa}{\rho_1}, \\ \frac{d\beta_\kappa}{ds} &= -\frac{\alpha_\kappa}{\rho_1} - \frac{\gamma_\kappa}{H}, \\ \frac{d\gamma_\kappa}{ds} &= -\frac{\delta_\kappa}{\rho_3} + \frac{\beta_\kappa}{H}, \\ \frac{d\delta_\kappa}{ds} &= \frac{\gamma_\kappa}{\rho_3}, \end{aligned} \tag{20}$$

a set of equations which express the derivatives with respect to the arc of the sixteen direction cosines as linear functions of the cosines themselves; the coefficients being functions of the three curvatures of the curve considered. Formulæ (20) are to curves of triple curvature what the Serret-Frenet formulæ are to curves of double curvature.

From equations (20) follow immediately on squaring and adding

$$\left. \begin{aligned} \frac{d\alpha_\kappa^2}{ds^2} &= \frac{1}{\rho_1^2}, & \frac{d\gamma_\kappa^2}{ds^2} &= \frac{1}{\rho_2^2}, \\ \frac{d\beta_\kappa^2}{ds^2} &= \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{1}{\rho_3^2}, & \frac{d\delta_\kappa^2}{ds^2} &= \frac{1}{\rho_3^2}, \end{aligned} \right\} \tag{21}$$

the first, third and fourth of which we have already obtained.

Spherical Indicatrix.

11. Suppose, through a fixed point, we construct a tetrahedroid with axes parallel to those of the principal tetrahedroid at a point M of a curve L , and at a distance of 1 from the origin on each of the axes parallel to the tangent, principal binormal and trinormal, choose points. These points will describe spherical curves on a sphere of radius unity whose center is the fixed point. We will call the curves σ , τ , k so described, the spherical indicatrix of the tangent, the spherical indicatrix of the principal binormal and the spherical indicatrix of the trinormal respectively.

By means of certain properties of these curves σ , τ , k obtained by using the kinematical equations (15), we can define the positive directions of the principal normal, principal binormal and trinormal, having given a positive direction on the tangent. For any point on σ equations (15) and (19), together with $p_{13} = p_{14} = p_{24} = 0$, give

$$V_x = 0, \quad V_y = \frac{1}{\rho_1}, \quad V_z = 0, \quad V_w = 0.$$

Then the tangent to σ is parallel to the principal normal to L , and the velocity of the point which traces out σ is exactly the first curvature. We choose, then, as the positive sense on the principal normal, the sense of the tangent to σ .

Imagine a man standing with his feet at the point M of L and the principal binormal passing through him lengthwise; the direction from his feet to his head will be called the positive direction of the principal binormal if he sees the x -axis become coincident with the y -axis after a rotation of 90° from left to right about the principal binormal.

If the line through the fixed point and parallel to the trinormal extends in both directions a distance unity from the fixed point, there will be two spherical curves k and k' traced on the unit sphere by points whose velocity is given by

$$V_x = 0, \quad V_y = 0, \quad V_z = \frac{1}{\rho_3}, \quad V_w = 0.$$

The velocity of the tracing points is therefore $\frac{1}{\rho_3}$ and the tangents to k and k' are parallel to the principal binormal of L . We shall speak of that one of the curves k and k' as the spherical indicatrix of the trinormal which has as its sense the positive sense along the principal binormal. Then the positive sense along the trinormal, will be from the fixed point to the spherical indicatrix of the trinormal.

Then, given a positive sense along the tangent to a curve L at a point M , the positive directions along the three other principal lines are determined.

For any point on τ the equations (15) and (19) give

$$V_x = 0, \quad V_y = -p_{23} = \sqrt{\frac{1}{\rho_2^2} - \frac{1}{\rho_3^2}}, \quad V_z = 0, \quad V_w = p_{34} = -\frac{1}{\rho_3},$$

which shows that the motion is in the plane yw and the velocity of the tracing point is the second curvature.

Similarly the point at a distance of 1 along the principal normal traces a curve Γ with a velocity given by

$$V^2 = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{1}{\rho_3^2},$$

What precedes shows that *the curves σ , τ , k are traced by points whose velocities are respectively the first, second and third curvatures of the curve considered, while the square of the velocity of the point tracing Γ is the sum of the squares of the velocity along σ and τ minus the square of the velocity along k .*

Principal directions and three curvatures at a point in terms of the coordinates of the point.

12. The osculating hyperplane at any point M of a curve of triple curvature may be defined as a hyperplane such that the distance to it from any point of the line infinitely near M is an infinitesimal of at least the fourth order. Being so defined we can get its equation

$$A(X-x) + B(Y-y) + C(Z-z) + D(W-w) = 0. \quad A = \begin{vmatrix} y' & z' & w' \\ y'' & z'' & w'' \\ y''' & z''' & w''' \end{vmatrix} \quad (22)$$

by a method exactly similar to that used in getting the equation of the osculating plane of a curve of double curvature.

When the arc is taken as the independent variable

$$\alpha_1 = \frac{dx}{ds} = x', \quad \alpha_2 = y', \quad \alpha_3 = z', \quad \alpha_4 = w'. \quad (23)$$

Also if M be a point on L and m be the corresponding point on the spherical indicatrix of the tangents to L , then, since the tangent at m is parallel to the

principal normal at M , the cosines of the principal normal to L at M are given by

$$\beta_1 = \frac{d\alpha_1}{d\sigma}, \dots, \beta_4 = \frac{d\alpha_4}{d\sigma},$$

where σ is the arc of the spherical indicatrix. Remembering that $\frac{d\sigma}{ds} = \frac{1}{\rho_1}$, we have

$$\beta_1 = \rho_1 \frac{d\alpha_1}{ds} = \rho_1 x'', \dots \beta_4 = \rho_1 w''. \quad (24)$$

The direction cosines of the trinormal being proportional to those of the perpendicular to the osculating plane, we have

$$\delta_1 = \frac{A}{\Delta}, \dots, \delta_4 = \frac{D}{\Delta}, \Delta^2 = A^2 + B^2 + C^2 + D^2. \quad (25)$$

If m' be the point on the spherical indicatrix of the trinormals corresponding to M , then, since the tangent at m' is parallel to the principal binormal of L ,

$$\gamma_1 = \frac{d\delta_1}{d\sigma}, \dots, \gamma_4 = \frac{d\delta_4}{d\sigma},$$

$d\sigma$ being the arc of the spherical indicatrix. Since $\frac{d\sigma}{ds} = \frac{1}{\rho_3}$

$$\gamma_1 = \rho_3 \frac{d\delta_1}{ds}, \dots, \gamma_4 = \rho_3 \frac{d\delta_4}{ds}. \quad (26)$$

If we notice that the determinant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix} = 1,$$

each element is exactly equal to its minor, we will have

$$\gamma_1 = \begin{vmatrix} \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \\ \alpha_4 & \beta_4 & \delta_4 \end{vmatrix}, \text{ etc}, \quad (27)$$

Squaring and adding equations (24) we find an expression for the first curvature in terms of the coordinates of the point;

$$\frac{1}{\rho_1^2} = x''^2 + y''^2 + z''^2 + w''^2. \quad (28)$$

Squaring and adding (26), we find again

$$\frac{1}{\rho_3^2} = \left(\frac{d\delta_1}{ds}\right)^2 + \left(\frac{d\delta_2}{ds}\right)^2 + \left(\frac{d\delta_3}{ds}\right)^2 + \left(\frac{d\delta_4}{ds}\right)^2, \quad (29)$$

which, combined with (25) gives the third curvature in terms of the coordinates of the point.

The expression for the second curvature in terms of the coordinates of the point may be obtained by using

$$\frac{1}{\rho_2^2} = \frac{d\gamma_1^2 + d\gamma_2^2 + d\gamma_3^2 + d\gamma_4^2}{ds^2},$$

together with (26), (25) and (29).

The three curvatures being found in terms of the coordinates of M , we can, by substitution in (23), (24), (27) and (25), find $\alpha_1, \dots, \delta_4$ in terms of the same coordinates.

Position of a curve L.

13. The positive sense on the principal binormal being defined as above, the center of first curvature is on the concave side of the curve. For, if P and P_1 are two infinitely near points on the curve, the projection of the chord PP_1 on the principal normal is

$$K = \beta_1(x_1 - x) + \beta_2(y_1 - y) + \beta_3(z_1 - z) + \beta_4(w_1 - w).$$

If the equations of the curve are

$$x = f_1(s), \quad y = f_2(s), \quad \dots,$$

then $x_1 = f_1(s + h), \quad y_1 = f_2(s + h), \quad \dots,$

and $x_1 = f_1(s) + hf'_1(s) + \frac{h^2}{2}f''_1(s) + \dots$

But $f_1(s) = x, \quad f'_1(s) = \alpha_1, \quad f''_1(s) = \frac{d\alpha_1}{ds} = \frac{\beta_1}{\rho_1}, \quad \dots,$

then $x_1 - x = h\alpha_1 + \frac{h^2\beta}{2\rho_1} + \frac{h^3}{6}(\beta_1\phi' - \alpha_1\phi^2 - \phi\psi\gamma_1) + \frac{h^4}{24}(\dots) + \dots$

where $\phi = \frac{1}{\rho_2}, \quad \psi = \frac{1}{H}, \quad \chi = \frac{1}{\rho_3}.$

Similar expressions may be obtained for $y_1 - y, z_1 - z, w_1 - w$ by advancing the

subscripts on the direction cosines. Substituting in the expression for K , we have

$$K^2 = \frac{h^2}{2\rho_1} + \dots$$

which is positive, whether h be — or +, provided it be infinitesimal. The projection K being positive, the center of curvature must be on the concave side of the curve. *The curve is entirely on one side of its tangent.*

The projection of PP_1 on the principal binormal is given by

$$\begin{aligned} K_1 &= \gamma_1(x_1 - x) + \gamma_2(y_1 - y) + \dots \\ &= -\frac{h^3\phi\psi}{6} + \dots; \end{aligned}$$

that is, according as h is + or —, the projection PP_1 on the principal binormal is — or +. *The curve, therefore, crosses its osculating plane.*

The projection K_2 on the trinormal is given by

$$\begin{aligned} K_2 &= \delta_1(x_1 - x) + \delta_2(y_1 - y) + \dots \\ &= \frac{h^4\phi\psi\chi}{24} + \dots; \end{aligned}$$

that is, whether h be + or —, *the curve is always on the same side of its osculating hyperplane.* Conf. Lovett, *Amer. Jour.*, vol. XXII, p. 226 et seq.

Coordinates of a point in terms of the arc.

14. To express the coordinates of any point of a curve L in terms of the arc, we let O be a point of L and M a point near it. Take O as the origin and construct the usual tetrahedroid. The coordinates of M are

$$x = f_1(s), \quad y = f_2(s), \quad \dots,$$

which, developed by McLaurin's formula, give

$$x = sf'_1(0) + \frac{s^3}{2} f''_1(0) + \frac{s^5}{6} f'''_1(0) + \dots$$

Substituting and using the abbreviations Φ, Ψ, X as in the previous section, we find

$$\left. \begin{aligned} x &= s - \alpha_1 \Phi^2 s^3 + \dots, \\ y &= \frac{\Phi}{2} s^2 + \frac{\Phi'}{6} s^3 + \dots, \\ z &= -\frac{\Phi\Psi}{6} s^3 + \dots, \\ w &= \frac{\Phi\Psi\chi}{24} s^4 + \dots \end{aligned} \right\} \quad (30)$$

it being understood that the values of Φ, Ψ, X are taken for $s = 0$. These formulæ also make it evident that the curve does not, in general, cross its osculating hyperplane.

It is easy to show by means of (30) that the difference between an infinitesimal arc and its chord is of the third order with respect to the arc.

The osculating hypersphere.

15. An equation of the form

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + (w - w_1)^2 = r^2$$

will be said to represent a hypersphere because of the similarity of its form to that of the equation of a sphere in ordinary space. The conditions for contact of order n between a curve of triple curvature and a hypersurface are like those for contact of a curve of double curvature and a surface in ordinary space, if the definition of contact of order n is:

When a curve L and a hypersurface S have in common a simple point P we say that they have at that point a contact of order n if, for every point Q infinitely near P on L , we can make correspond on S a point Q' such that the distance QQ' be infinitesimal of order $n + 1$ with respect to the chord PQ .

The hypersphere which, at a point $M(x, y, z, w)$ of a curve L , has a contact of highest order possible with L , is called the osculating hypersphere of L at M . Since the equation of the hypersphere contains five parameters, the contact must be of the fourth order if the hypersphere is to osculate the curve, and if we call X, Y, Z, W the coordinates of the center of the osculating hypersphere and R its radius, we have

$$\begin{aligned} \phi(s) &= \Sigma (x - X)^2 - R^2 = 0, \\ \frac{1}{2} \phi'(s) &= \Sigma (x - X) \alpha_1 = 0, \\ \phi''(s) &= \phi'''(s) = \phi^{IV}(s) = 0. \end{aligned}$$

The second of these is the equation of the normal hyperplane at M . If we differentiate the second equation and use the first equation (20); differentiate again and use the second of (20), and, finally, differentiate again and use the third equation (20), we have, for the determination of the radius R , the equations

$$\left. \begin{aligned} (x - X) \alpha_1 + (y - Y) \alpha_2 + (z - Z) \alpha_3 + (w - W) \alpha_4 &= 0, \\ (x - X) \beta_1 + (y - Y) \beta_2 + (z - Z) \beta_3 + (w - W) \beta_4 &= -\rho_1, \\ (x - X) \gamma_1 + (y - Y) \gamma_2 + (z - Z) \gamma_3 + (w - W) \gamma_4 &= H \frac{d\rho_1}{ds}, \\ (x - X) \delta_1 + (y - Y) \delta_2 + (z - Z) \delta_3 + (w - W) \delta_4 \\ &= -\left(\frac{\rho_1 \rho_3}{H} + \rho_3 \frac{d}{ds} H \frac{d\rho_1}{ds}\right). \end{aligned} \right\} \quad (31)$$

Squaring and adding these four equations we find

$$\begin{aligned} (x - X)^2 + (y - Y)^2 + (z - Z)^2 + (w - W)^2 &= R^2 \\ &= \rho_1^2 + H^2 \frac{d\rho_1^2}{ds^2} + \left(\frac{\rho_1 \rho_3}{H} + \rho_3 \frac{d}{ds} H \frac{d\rho_1}{ds}\right)^2, \end{aligned} \quad (32)$$

which is the expression sought for the radius of the osculating hypersphere at any point of a curve of triple curvature. The first two terms form the expression for the radius of the osculating sphere at a point of a curve of double curvature.

*Locus of centres of osculating hyperspheres.**

16, The consecutive normal hyperplanes of a curve L of triple curvature envelop a curved hypersurface, S_4 , which is called the *polar hypersurface* of L . Two consecutive normal hyperplanes have in common a plane called the *polar plane* of L . These polar planes envelop a developable surface S_3 called the *surface of regression* of S_4 . Three consecutive normal hyperplanes have in common a right line called the *polar line* of L , and these polar lines envelop a curve called the *edge of regression* of S_3 or the *line of regression* of S_4 . Then, with these conventions, what precedes shows that the center of the osculating hypersphere is at the point where the polar line touches its envelope, and its locus is the edge of regression of the polar hypersurface.

The coordinates of the center of the osculating hypersphere may be found by

* Pirondini, "Giornali di Mathematiche"; loc. cit.

solving equations (31) for $x = X$, $y = Y$, $z = Z$, $w = W$, a process which gives

$$\left. \begin{aligned} X &= x + \beta_1 \rho_1 - \gamma_1 H \frac{d\rho_1}{ds} + \delta_1 K, & Z &= z + \beta_3 \rho_1 - \gamma_3 H \frac{d\rho_1}{ds} + \delta_3 K, \\ Y &= y + \beta_2 \rho_1 - \gamma_2 H \frac{d\rho_1}{ds} + \delta_2 K, & W &= w + \beta_4 \rho_1 - \gamma_4 H \frac{d\rho_1}{ds} + \delta_4 K, \\ & & K &= \frac{\rho_1 \rho_3}{H} + H \rho_3 \frac{d^2 \rho_1}{ds^2} + \rho_3 \frac{d\rho_1}{ds} \frac{dH}{ds}. \end{aligned} \right\} \quad (33)$$

If equations (33) be differentiated and the results simplified by the use of (20), the resulting formulae will give the direction cosines of the tangent to the curve traced out by the center of the osculating hypersphere ;

$$\frac{dX}{ds} = \delta_1 \left(\frac{H}{\rho_3} \frac{d\rho_1}{ds} + \frac{dK}{ds} \right), \quad \frac{dY}{ds} = \delta_2 \left(\frac{H}{\rho_3} \frac{d\rho_1}{ds} + \frac{dK}{ds} \right), \quad \dots \quad (34)$$

The second members of these equations containing $\delta_1, \delta_2, \delta_3, \delta_4$ as factors, it follows that *the tangent to the locus of the centers of the osculating hyperspheres of a curve L of triple curvature is parallel to the trinormal at the corresponding point of L .*

If L be a curve of triple curvature and L_0 , be the locus of the centers of its osculating hyperspheres, the osculating hyperplane of L is parallel to the normal hyperplane of L_0 , since the trinormal of L is parallel to the tangent of L_0 . Also, the osculating hyperplane of L_0 having the polar line of L as its characteristic coincides with the normal hyperplane of L and, consequently, the trinormal of L , is parallel to the tangent of L . The principal binormal of L_0 lies in the osculating hyperplane of L_0 perpendicular to two consecutive tangents of L_0 ; that is, it lies in the normal hyperplane of L perpendicular to two consecutive trinormals of L . Therefore it is parallel to the principal normal. It follows immediately that the principal normal of L_0 is parallel to the principal binormal of L . We have then the theorem ;

The tangent, principal normal, principal binormal and trinormal of a curve L of triple curvature are respectively parallel to the trinormal, principal binormal, principal normal and tangent of the locus L_0 of the centers of the osculating hyperspheres of L .

17. The tangent to L_0 at any point being parallel to the trinormal of L at the corresponding point, and the trinormal of L_0 being parallel to the tangent of L , it follows that the angle of two tangents of L_0 is equal to the angle of two

osculating hyperplanes of L ; and the angle of two osculating hyperplanes of L_0 is equal to the angle of two tangents to L . We have therefore

$$\frac{\text{angle of two tangents of } L_0}{ds_0} = \frac{1}{\rho_1^0},$$

$$\frac{\text{angle of two osculating hyperplanes of } L}{ds} = \frac{1}{\rho_3},$$

and, consequently,

$$\frac{ds_0}{\rho_1^0} = \frac{ds}{\rho_3}.$$

In like manner,

$$\frac{\text{angle of two osculating hyperplanes of } L_0}{ds_0} = \frac{1}{\rho_3^0},$$

$$\frac{\text{angle of two tangents of } L}{ds} = \frac{1}{\rho_1},$$

whence,

$$\frac{ds_0}{\rho_3^0} = \frac{ds}{\rho_1},$$

where ds^0 , ρ_1^0 , ρ_3^0 denote the arc, radius of first curvature and radius of third curvature of L_0 . Comparing these results, we find immediately the relation

$$\rho_1 \rho_1^0 = \rho_3 \rho_3^0.$$

We can also find ρ_1^0 and ρ_3^0 ; for, from (26), we have

$$\frac{ds_0}{ds} = \frac{H}{\rho_3} \frac{d\rho_1}{ds} + \frac{dK}{ds} \equiv V.$$

Consequently, *given the three curvatures of a curve of triple curvature, we can find immediately the first and third curvatures of the locus of the centers of its osculating hyperspheres by means of the equations*

$$\rho_1^0 = \rho_3 V, \quad \rho_3^0 = \rho_1 V;$$

V being given by the equation above.

18. *The osculating hypersphere and the osculating hyperplane at a point M of L intersect in an ordinary sphere which is the ordinary osculating sphere at M , and is the characteristic of the osculating hypersphere at M .*

For the characteristic of the osculating hypersphere at M is given by

$$\begin{aligned} S(x - X)^2 - R^2 &= 0, \\ S(x - X) \frac{dX}{ds} + R \frac{dR}{ds} &= 0, \end{aligned}$$

the latter being the equation of the hyperplane determining the characteristic. By using (34), this last equation may be written

$$[S(x - X)\delta_1] \left[\frac{H}{\rho_3} \frac{d\rho_1}{ds} + \frac{dK}{ds} \right] + R \frac{dR}{ds} = 0, \quad (35)$$

which shows that the hyperplane determining the characteristic is parallel to the osculating hyperplane. It also passes through the point M ; for, substituting from (33) the values of X , Y , Z , W , we find

$$R \left[\frac{H}{\rho_1} \frac{d\rho_1}{ds} + \frac{dK}{ds} \right] - R \frac{dR}{ds} = 0,$$

which is seen to be verified if we differentiate (32).

Hyperspherical curves.

19. Any curve of triple curvature which is situated on a hypersphere is called a hyperspherical curve. Since the osculating hypersphere of such a curve is the hypersphere on which it lies, we have necessarily at every point of a hyperspherical curve

$$\rho_1^2 + H^2 \frac{d\rho_1^2}{ds^2} + K^2 = \text{const.} \quad (36)$$

Conversely, if at every point of a curve of triple curvature the radius of the osculating hypersphere is constant, the curve is a hyperspherical curve. For, noticing that

$$\frac{K}{\rho_3} - \frac{d}{ds} \left(H \frac{d\rho_1}{ds} \right) - \frac{\rho_1}{H} = 0,$$

and, therefore,

$$\frac{H}{\rho_3} \frac{d\rho_1}{ds} = \frac{H}{K} \frac{d}{ds} \left(H \frac{d\rho_1}{ds} \right) + \frac{\rho_1}{K} \frac{d\rho_1}{ds}$$

we can get from equations (26),

$$\begin{aligned} \frac{1}{\delta_1} \frac{dX}{ds} &= \frac{1}{\delta_2} \frac{dY}{ds} = \frac{1}{\delta_3} \frac{dZ}{ds} = \frac{1}{\delta_4} \frac{dW}{ds} \\ &= \frac{1}{K} \left(H \frac{d\rho_1}{ds} \right) \frac{d}{ds} \left(H \frac{d\rho_1}{ds} \right) + \rho_1 \frac{d\rho_1}{ds} + K \frac{dK}{ds}, \end{aligned}$$

the last member of which is seen to be zero if we differentiate (36). Consequently, the center of the osculating hypersphere is fixed; and, since by hypothesis its radius is constant, it is the same at all points of the curve, and the curve itself is hyperspherical. Hence, the necessary and sufficient condition that a curve of triple curvature be hyperspherical is that the radius of its osculating hypersphere be constant. The equation expressing this condition is (36), but this, on account of (34), may be replaced by

$$\frac{H}{\rho} \frac{d\rho_1}{ds} + \frac{dK}{ds} = 0,$$

which is characteristic of hyperspherical curves.

Involutes of curves of triple curvature.

20. If a curve L be such that each of its tangents is normal to another curve L_2 , we say that L_2 is the involute of L , and that L is the evolute of L_2 .

At any point of L , construct the usual tetrahedroid. According to the above definition, the involute L_2 is traced out by such a point on the axis of x of T that its locus is always perpendicular to the x -axis. Let 1, 0, 0, 0 be the coordinates of this point referred to the moving axes. By using (15), we find the projections of its velocity to be

$$\left. \begin{aligned} V_x &= 1 + \frac{dl}{ds}, \\ V_y &= \frac{l}{\rho_1}, \\ V_z &= V_w = 0. \end{aligned} \right\} \quad (37)$$

Expressing that the velocity is always perpendicular to the x -axis, we have

$$dl + ds = 0 \text{ or } l + s = \text{const.}, \quad (38)$$

a property which holds for curves of double curvature.

Equations (37) also show that the direction of the velocity of the point describing the involute is that of the principal normal of the evolute, and its magnitude is l/ρ_1 .

Comparing this with what precedes, we see that the tangent to the involute, the principal normal to the evolute and the tangent to the spherical indicatrix of the evolutes, are parallel lines. Also, the velocity of the point describing the involute is l times that of the corresponding point on the spherical indicatrix mentioned.

If we define the rectifying hyperplane at any point of a curve of triple curvature as the hyperplane perpendicular to the principal normal at that point, we can say that the rectifying hyperplane of the evolute is the normal hyperplane of the involute.

Evolutes of a curve L .

21. Consider the tetrahedroid T attached in the usual manner to the curve L at a point $M(x, y, z, w)$. The evolute of L must be traced out by such a point M_1 in the normal hyperplane of L that the tangent to its locus must always pass through M . Let the coordinates of M_1 with respect to the moving axes be $0, y_1, z_1, w_1$; then, using (15) we find the projections of its velocity to be

$$\left. \begin{aligned} V_x &= 1 - p_{12}y_1, \\ V_y &= \frac{dy_1}{ds} - p_{23}z_1, \\ V_z &= \frac{dz_1}{ds} + p_{23}y_1 - p_{34}w_1, \\ V_w &= \frac{dw_1}{ds} + p_{34}z_1. \end{aligned} \right\} \quad (37)$$

The first of these gives immediately

$$y_1 = \rho_1;$$

that is, the point M_1 is on the polar line and consequently *all the evolutes of a curve of triple curvature lie on the surface of regression of the polar hypersurface.*

If the tangent to the locus of M_1 is to pass through M we must have

$$\frac{\frac{dy_1}{ds} - p_{23}z_1}{y_1} = \frac{\frac{dz_1}{ds} + p_{23}y_1 - p_{34}w_1}{z_1} = \frac{\frac{dw_1}{ds} + p_{34}z_1}{w_1},$$

$$\left. \begin{aligned} y_1 dz_1 - z_1 dy_1 + p_{23}(y_1^2 + z_1^2) ds - p_{34}w_1 y_1 ds &= 0, \\ z_1 dw_1 - w_1 dz_1 + p_{34}(w_1^2 + z_1^2) ds - p_{23}w_1 y_1 ds &= 0. \end{aligned} \right\} \quad (40)$$

Projecting the first of these on the plane of yz gives

$$d \cdot \arctan \frac{z_1}{y_1} = \frac{y_1 dz_1 - z_1 dy_1}{y_1^2 + z_1^2} = -p_{23} ds,$$

which may be written

$$d \cdot \arctan \frac{z_1}{y_1} = \frac{ds}{H}.$$

On integrating, we find

$$\phi_1 = \arctan \frac{z_1}{y_1} = \int \frac{ds}{H}, \quad (41)$$

and, therefore,

$$z_1 = y_1 \tan \phi_1 = \rho_1 \tan \phi_1.$$

Similarly, from the second equation (40) we find

$$\phi_2 = \arctan \frac{w_1}{z_1} = \int \frac{ds}{\rho_3}, \quad (42)$$

and hence,

$$w_1 = z_1 \tan \phi_2 = \rho_1 \tan \phi_1 \tan \phi_2.$$

If we denote the coordinates of M_1 with respect to the fixed axes by X_1, Y_1, Z_1, W_1 , we have

$$\begin{aligned} X_1 &= x + \beta_1 \rho_1 + \gamma_1 \rho_1 \tan \phi_1 + \delta_1 \rho_1 \tan \phi_1 \tan \phi_2, \\ Y_1 &= y + \beta_2 \rho_1 + \gamma_2 \rho_1 \tan \phi_1 + \delta_2 \rho_1 \tan \phi_1 \tan \phi_2, \\ &\dots\dots\dots \end{aligned}$$

which, when $\rho_3 = \infty$, readily reduce to the corresponding formulæ for ordinary space.

Equations (41) and (42) show that ϕ_1 and ϕ_2 contain additively an arbitrary constant. If, then, we are given a family of normals touching a known evolute, and if we rotate each normal about the tangent at its point of incidence in such a way that ϕ_1 and ϕ_2 are increased by fixed amounts, we get a second family of normals touching a second evolute. Any curve of triple curvature has, therefore ∞^2 evolutes.

These same formulæ show that the principal normals of a curve of triple curvature cannot have an envelope ; for, in order that ϕ_1 and ϕ_2 be zero, we must have

$$\rho_3 = \rho_2 = \infty,$$

and the curve would reduce to a plane curve.

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